Comment on “Network analysis of the state space of discrete dynamical systems”

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Jan 10, 2017 - Feb 15, 2017

This paper discusses the letter entitled “Network analysis of the state space of discrete dynamical systems” by A. Shreim et al. [Physical Review Letters, 98, 198701 (2007)]. We found that some theoretical analyses are wrong and the proposed indicators based on parameters of phase network can not discriminate two parameters of the state-mapping network cannot discriminate the dynamical complexity of the discrete dynamical systems composed by of a 1-D Cellular Automata.

Keywords: Chaotic dynamics; Cellular Automata; complex network; state-mapping network.

1. Introduction

Methodology of complex networks has been used as an important tool to investigate the dynamics of some discrete systems and even chaotic cryptanalysis [Liou & Chen, 2012; Kayama, 2012; Sanchez & Rodriguez, 2015; Wang et al., 2016; Xie et al., 2017]. The general framework of the analyses can be summarized as four stages: selection of nodes; establishment of linkage between any pair of nodes; definition of some characteristic parameters of the obtained network; comparison with other recognized (convincing) metrics. Interestingly, some subtle dynamical properties of the underlying discrete systems were disclosed from the perspective of complex networks [Wiensche & Lesser, 1992; Kyriakopoulos & Thurner, 2007; Luque et al., 2011].

In paper [Shreim et al., 2007], the authors analyzed the state space state-mapping network of discrete dynamical systems composed by of a 1-D cellular automata (CA) with rules involving two colors and nearest neighbors, and the nearest neighbors. The studied CA sets the binary value $s_i^{t+1}$ of site $i$ in the $(t+1)$th
iteration as \( R(s_{i+1}^{l} s_{i}^{l+1}) \) counted at the previous one, where \( R \) denotes the CA rule with an identifying number \( R(000)^2 + R(001)^2 + R(010)^2 + R(011)^2 + R(100)^2 + R(101)^2 + R(110)^2 + R(111)^2 \). Then, two characteristic parameters of the built directed network were adopted: the largest in-degree and the path diversity measuring fluctuations among different paths connecting the nodes corresponding to the transient states with zero in-degree and that corresponding to attractors. Based on observation results on only ten rules, the authors claimed that “the scaling and distribution of in-degrees and the path diversity give a good indication of dynamical complexity”. Concretely, More precisely, the statement means that the co-appearance of nontrivial scaling in both the hub size of both in-degree distribution (or the hub sizes) and the path diversity of the phase networks a state-mapping network with its system size can separate simple dynamics from the more complex ones found in CA falling in Wolfram’s class III, classes III and IV. This comment-paper is to point out that the analytic results for rule 4 given in [Shreim et al., 2007] are wrong and the proposed indicators cannot discriminate the dynamical complexity of cellular automata CA at all.

2. **Network Discussion on network analysis of the state space of CA with rule 4**

In [Shreim et al., 2007], there are some obvious defects on discrepancies on the estimation of in-degree distribution of phase-the state-mapping network corresponding to rule 4. The authors adopted \( a \lambda_m \) as \( a \cdot \lambda_m \) as an approximate value of \( w(m) \). Unfortunately, the estimation error is:

\[
w(m_i) = e_3 [T^{(0)}]^{m_i} e_2^T,
\]

where \( \lambda \) is the largest eigenvalue of \( T^{(0)} \), a fixed binary matrix facilitating counting the in-degree of any state-mapping of the state-mapping network of CA with rule 4, \( e_2 = (0, 1, 0, 0) \) and \( e_3 = (0, 0, 1, 0) \), and \( m_i \) is the number of ‘0’s following the \( i \)th ‘1’ in the counted state. However, the fact is that the estimation errors are accumulated exponentially in the multiplication calculation of multiplications in

\[
k_n = \prod_{i=1}^{n} w(m_i) = \prod_{i=1}^{n} (a \lambda_m)^i.
\]

The authors of [Shreim et al., 2007] calculate \( \Omega(n) \), the number of states with \( n \) isolated 1’s and no pairs but no pairs of ‘11’, as obtaining \( C(L - n, n) + C(L - n - 1, n - 1) \). Actually, \( \Omega(n) = C(L - n + 1, n) = C(L - n, n) + C(L - n, n - 1) \). As for rule 4, every non-isolated node satisfies \( L - n + 1 \geq n \). So, Eq. (7) in [Shreim et al., 2007] outputs complex numbers:

\[
y \approx -1 - \log_2 \left[ \frac{(1 - \epsilon)^{1-\epsilon}}{\epsilon^\epsilon (1 - 2\epsilon)^{1-2\epsilon}} \right],
\]

outputs complex numbers as \( (1 - 2\epsilon) \leq 0 \) when \( L \leq 2n \leq L + 1 \), where \( \epsilon \equiv n/L \). The cascaded cascading effect of these defects make errors implies that Eq. (7) cannot accurately approximate the distribution of the largest in-degree values. In addition, the scope of scale of the Y-axis of in Fig. 4 in [Shreim et al., 2007] should be \([-1, 0]\), not \([-2, 0]\), which can be deduced by from \( 1/N \leq P(k) \leq 1 \) and \(-1 \leq \log P(k) / \log N \leq 0 \). We re-calculated the in-degree distribution functions for three rules under three systems with three system sizes in Fig. 1 and re-drew Eq. (7) at its left part as-by a bold black curve, which support confirm our findings.

As for As to the 1-D CA under study, the 256 elementary rules are reduced to 88 independent rules under some transformation ones under some transformations [Li & Packard, 1990, Page 294]. However, only ten independent rules were selected to verify the statement in [Shreim et al., 2007]. We calculated the change slope of the largest in-degree value and path diversity with respect to the CA size for every independent rule and now depict the results in Fig. 2, which is divided into four panels according to Wolfram’s classification. The concrete classification specification is referred to the detailed classification is referred to [Wolfram, 2002, Page 231], [Li & Packard, 1990, Table 1], and [Zenil & Villarreal-Zapata, 2013]
Fig. 1. In-degree distribution functions for three rules under different system sizes.

Table 3]. From each panel, no general pattern can be observed. Furthermore, we can see that both the two observed values are very close among rule to each other among rules 54, 62, 110, and 182, which belonging to three different classes. More counterexamples can be found to verify given to disprove inefficiency of the main statement of [Shreim et al., 2007] from even just based on Fig. 2.

3. Conclusion

Using the example given in [Shreim et al., 2007] as a counterexample, we This paper briefly reported three key errors in the deduction process of estimating an in-degree distribution function in [Shreim et al., 2007]. With the help of Wolfram’s classification on CA, we further demonstrated that the in-degree distribution and the path diversity of phase network cannot be used to measure the dynamical complexity of the systems corresponding underlying system formed by a 1-D CA. Justification for measuring complexities of general discrete dynamical systems using statistical characteristics of their phase state-mapping networks needs further investigation [Liou & Chen, 2012; Kayama, 2012; Wang et al., 2016]. More detailed analysis on the phase state-mapping networks of 1-D CAs will be reported in a forthcoming paper.

Acknowledgement

This research was supported by the Natural Science Foundation of China (No. 61532020, 61611530549) and Scientific Research Fund of Hunan Provincial Education Department (No. 15A186).

References

Fig. 2. Distribution of the two observables defined in [Shreim et al., 2007] with respect to the rule numbers.